Reply to 'Comment on "On the general solution for the modified Emden type equation $\ddot{x} + \alpha x \dot{x} + \beta x^3 = 0$"'

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REPLY

Reply to 'Comment on "On the general solution for the modified Emden type equation $\ddot{x} + \alpha x x + \beta x^3 = 0$"'

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Abstract
Explicit/implicit solutions for general parametric values of the modified Emden equation and its generalizations were obtained recently by us using a modified Prelle–Singer approach coupled with a Hamiltonian formalism. In his comment, Iacono has suggested an interesting alternate approach. Here we point out the limitations and incompleteness of the approach as well as some additional elementary approaches to the above equations.

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In [1] we have obtained explicit solutions of the modified Emden equation (MEE), $\ddot{x} + \alpha x\dot{x} + \beta x^3 = 0$, first by writing down the time-independent first integrals obtained through a modified Prelle–Singer approach and then developing a Hamiltonian formalism and integrating the resultant canonical equations. We have also pointed out that in the literature only for two parametric choices, namely (i) $\beta = \alpha^2/9$ and (ii) $\beta = -\alpha^2$, explicit solutions were reported earlier. It was also noted that though the system admits two parameter Lie point symmetries, which guarantee the integrability of the system, neither explicit integration was performed nor an explicit solution was obtained for general parametric values of α and β for this physically important system in spite of the existence of a large literature available on it (see, for example, [1–19] in [1]). In our paper [1], we have succeeded to do this.

In his comment [2] on our paper, Iacono points out the interesting fact that the time-independent integrals of the MEE reported in [1] can be obtained in an alternate way by using the connection between the MEE and certain first-order Abel's equation (AE). Following this, he suggests two ways to obtain the solution of the MEE: either (1) to follow our Hamiltonian approach [1] or (2) to obtain an implicit solution directly. There are several comments in order.

(i) It is not surprising to identify alternate methods to obtain integrals and solutions of nonlinear differential equations once their forms are deduced by a specific method. This is indeed the case for all soliton equations: inverse scattering transform, Bäcklund transformation, Hirota bilinearization, group theoretical, etc methods all give similar results. An equation solvable by one method is always found to be solvable by the other methods. It is very subjective to claim that one method is simpler than the other. Each one has its own advantages and disadvantages. In the present case, the modified Prelle–Singer procedure which we have advocated in our papers not only works for the MEE-type equations but also for any second-order integrable equation, and it has also been extended to third order [3], *n*th order [4] and system of ordinary differential equations [5], while the method suggested in [2] has only a limited application to Liénard-type second-order equations satisfying relation (5) of [2].

Further, there are other even more direct ways of finding the integrals of the MEE as pointed to us by Janzen recently [6]. Rewriting the MEE as

$$\ddot{u} + 4au\dot{u} + 2u^3 = 0, \qquad x = \sqrt{\frac{2}{\beta}}u, \qquad a = \frac{\alpha}{\sqrt{8\beta}}, \tag{1}$$

equation (1) can be re-expressed as

$$(\dot{u} + \gamma u^2)^{\cdot} + (4a - 2\gamma)u\left(\dot{u} + \frac{u^2}{2a - \gamma}\right) = 0. \tag{2}$$

Defining $1/(2a - \gamma) = \gamma$ or $\gamma^2 - 2a\gamma + 1 = 0$, so that the two roots become $\gamma_1 = a + \sqrt{a^2 - 1}$ and $\gamma_2 = a - \sqrt{a^2 - 1}$, respectively, equation (2) can be rewritten as

$$\begin{aligned}(\dot{u} + \gamma_1 u^2)^{\cdot} + 2\gamma_2 u(\dot{u} + \gamma_1 u^2) &= 0,\\ (\dot{u} + \gamma_2 u^2)^{\cdot} + 2\gamma_1 u(\dot{u} + \gamma_2 u^2) &= 0.\end{aligned} \tag{3}$$

Then for general initial conditions, except for $\dot{u} + \gamma_i u^2 = 0, i = 1, 2$, one can integrate (3) to yield

$$I = \frac{(\dot{u} + \gamma_1 u)^{\gamma_1}}{(\dot{u} + \gamma_2 u)^{\gamma_2}}. \tag{4}$$

For suitable functional forms of I, one can then obtain the integrals and Hamiltonians given in [1]. Thus, it is clear that once the results of [1] are known then there are many ways of obtaining the same result.

Yet another simple way to find the integrals is as follows. Under the transformations $y = \int f(x)\,\mathrm{d}x$ and $\mathrm{d}z = f(x)\,\mathrm{d}t$, the damped harmonic oscillator equation,

$$y'' + \alpha y' + \beta y = 0, \qquad \left(' = \frac{\mathrm{d}}{\mathrm{d}z}\right), \tag{5}$$

can be transformed to the Liénard equation (LE) of the form

$$\ddot{x} + \alpha f(x)\dot{x} + \beta f(x)\int f(x)\,\mathrm{d}x = 0, \qquad \left(^{\cdot} = \frac{\mathrm{d}}{\mathrm{d}t}\right), \tag{6}$$

which is nothing but the integrable LE given in [2]. One can substitute the above transformation into the time-independent integrals of equation (5) given in [7] and obtain time-independent integrals for the integrable LE (6) for the three parametric regimes. If the integral $\int f(x)\,\mathrm{d}x$ is an invertible function of x, then from $y = \int f(x)\,\mathrm{d}x$ one can obtain the implicit solution of LE (6). Otherwise, one can use our Hamiltonian approach [1].

(ii) It may also be noted that though any function of the integral I may be taken as the Hamiltonian, only certain specific choices of these provide an identification of suitable canonically conjugate momenta and allow explicit integration; see, for example, appendix A of [7]. This is where the method advocated by us provides in a natural way the convenient forms of the Hamiltonian leading to explicit/implicit solutions.

(iii) The direct integration of the integrals suggested by Iacono [2] lead to the quadratures, vide equation (18) of [2],

$$t - t_0 = \frac{2}{\alpha}\exp\left[\frac{-c}{4}\right]\int \frac{\mathrm{d}w}{w^{\frac{1}{2}}\left(w+\frac{1}{2}\right)^{\frac{3}{2}}\exp\left[\frac{1}{4\left(w+\frac{1}{2}\right)}\right]}, \qquad \alpha^2 = 8\beta, \tag{7}$$

$$t - t_0 = \frac{1}{2\alpha c}\int \frac{\left(\frac{4\beta}{\alpha^2} - 3w - 6w^2\right)\left(w^2 + w + \frac{2\beta}{\alpha^2}\right)^{\frac{1}{4}}\mathrm{d}w}{w^{\frac{1}{2}}\left(w+\frac{1}{2}-A\right)^{1-\frac{1}{8A}}\left(w+\frac{1}{2}+A\right)^{\frac{1}{8A}-1}}, \qquad \alpha^2 > 8\beta, \tag{8}$$

$$t - t_0 = \frac{1}{\sqrt{2\alpha\exp[c]}}\int \frac{\exp\left(\frac{\alpha}{2\sqrt{8\beta-\alpha^2}}\tan^{-1}\left[\frac{2\alpha\left(w+\frac{1}{2}\right)}{\sqrt{8\beta-\alpha^2}}\right]\right)}{\left(w^2+w+\frac{2\beta}{\alpha^2}\right)^{\frac{3}{4}}}\,\mathrm{d}w, \qquad \alpha^2 < 8\beta, \tag{9}$$

where $w = \frac{\beta}{\alpha}\frac{x^2}{\dot{x}}$, $A = \frac{\sqrt{\alpha^2-8\beta}}{2\alpha}$ and c is an arbitrary integration constant. In contrast to what is claimed in [2], the integrals on the right-hand sides of the equations (7)–(9) cannot be explicitly carried out and so the explicit/implicit solutions obtained in our paper [1] cannot be obtained. We believe that it is again at this point that Hamiltonian approach developed in [1] is more profound and applicable not only to the present system but also to a larger class as well.

(iv) The same above assertions hold good for the additional comments made in [2] on the integrability and solutions of a system which we have discussed in a different journal [8]. It pertains to a general class of physically important nonlinear oscillator equations of the form $\ddot{x} + (k_1x^q + k_2)\dot{x} + k_3x^{2q+1} + k_4x^{q+1} + \lambda x = 0$, where k_i's, $i = 1, 2, 3, 4, \lambda$, and q are arbitrary parameters. Again following a systematic approach, we have identified several integrable cases, given their explicit forms of the first integrals and explicit solutions or proved Liouville integrability by identifying an appropriate time-independent Hamiltonian formalism. In the comment [2], Iacono claims that through a set of transformations on the dependent and independent variables, x and t, which are actually given in our papers [1, 8], some of the identified integrable cases of the above nonlinear oscillator equation are related to Abel's equation. Again, in this approach the crucial integrals cannot be carried out in any simple manner and in our opinion the alternate method suggested in [2] is incomplete for this equation also.

To conclude, we stand by our claims in our papers [1, 8] regarding the explicit integrability of the MEE and different new integrable systems in the generalized oscillator equation and that these results have never been reported in the literature. There is indeed an urgent need to identify and isolate such integrable systems and obtain their solutions for understanding the physical properties of the underlying systems. The alternative method suggested in the comment [2] requires further deeper analysis to conclude for which systems it can lead to really useful new information.

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